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Construction of Supersaturated Split-Plot Designs

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Abstract

We propose a combinatorial construction method for setting up informative experiments with both restricted randomisation and a large number of factors. The supersaturated split-plot designs (SSSPDs) are very useful in screening situations where the number of factors is larger than the number of available observations and several of these factors have levels that they are hard to change. The construction method is based on compound orthogonal arrays. We evaluate the constructed designs using an optimality criterion and we provide a lower bound for this criterion.

Keywords: compound orthogonal array, supersaturated split-plot design, optimality, lower bound.

1 Introduction

In quantitative work in any field of application, data collection issues are at least as important as data analysis. Haphazard experimentation can be very wasteful of resources (for an overview in design of experiments, see [5] and [4]).The supersaturated split-plot designs (SSSPDs) combine two very important classes of designs for screening situations. Firstly, the supersaturated designs (SSDs) is a large class of factorial designs which can be used for screening out the important factors from a large set of potentially active variables. They are designs with m factors and n observations, where $n \leq m$. In [7], an extensive literature review on the constructions and analysis methods of supersaturated designs is provided. Supersaturated experiments are usually designed assuming the treatments (combinations of factor levels) are completely randomised to the experimental units. However, just as with any designed experiment, there may be some structure in the experimental units; for example, if the units are sequential runs some factors may have to be changed less frequently than every run (e.g. factors whose levels are hard or costly to change). The split-plot designs are very effective in reducing the cost of an experiment in the presence of hard-to-change factors and/or of two-stage processes. There is a large body of published research on split-plot designs (e.g., [8], [1], [10]). Specifically, a significant amount of research focuses on fractional factorial split-plot designs. These designs are constructed by using regular or non-regular fractional factorial designs at the two-stage randomisation. Examples of construction methods, using regular designs, can be found in [2] and [6]. When non-regular fractions are considered, then the designs usually are found by considering orthogonal arrays (see [15], [14]). In this work, we use suitable fractional factorial designs as well, which they have the additional feature of more columns than rows, hence the constructed split-plot design is a supersaturated design at the same time.

The merging of these two classes of designs described above (SSDs and split-plot designs) is a relatively unexplored research area. The most relevant published work was done by [11]. The authors constructed a limited class of supersaturated split-plot designs based on Plackett-Burman designs and they used stepwise methods to analyse the data. Our method generalizes the method in [11] and it is based on compound orthogonal arrays (COAs). This class of orthogonal arrays (OAs) was first appeared in [12], while a construction method was presented in [13]. Later on in [9] a catalog of two-level COAs was provided.

In this work, we use the class of COAs in order to construct supersaturated split-plot designs with desired orthogonality properties between specific factors. In Section 2, we give some basic notations that we will use in the remainder of this paper and the definition of a COA. In Section 3, we are presenting the proposed construction method and the optimality criterion that will be used for the evaluation of these designs. A lower bound (LB) for this criterion is provided in Section 4, while some explanatory examples are given in Section 5. We end this paper with some conclusions.

2 Notations and Preliminaries

Consider a fractional factorial split-plot experiment involving m_1 whole-plot factors and m_2 subplot factors. Suppose all these $(m_1 + m_2)$ factors are at two levels. A typical level combination of the m_1 whole-plot factors will be denoted as $\boldsymbol{y}^{(1)} = y_1^{(1)} y_2^{(1)} \cdots y_{m_1}^{(1)}$, $y_j^{(1)} = 0, 1$, and $1 \leq j \leq m_1$. Let $V^{(1)}$ be the set of all such level combinations of m_1 whole-plot factors. Also, a typical level combination of the m_2 subplot factors will be denoted as $\boldsymbol{y}^{(2)} = y_1^{(2)} y_2^{(2)} \cdots y_{m_2}^{(2)}$, $y_j^{(2)} = 0, 1$, and $1 \leq j \leq m_2$ and the corresponding set of all such level combinations will be denoted as $V^{(2)}$. A typical level combination (or run) of the whole design, all the $(m_1 + m_2)$ factors taken together, will be denoted as $\boldsymbol{y} = \boldsymbol{y}^{(1)} \boldsymbol{y}^{(2)}$. The following definitions will be helpful in developing the rest of the paper.

Definition 1 *An orthogonal array $OA(n, m, s_1 \times s_2 \times \cdots \times s_m, t)$, having n rows, $m(\geq 2)$ columns, $s_1, \cdots, s_m$ symbols and strength t is an $n \times m$ array, with elements in the jth column from a set of s_j distinct symbols $(1 \leq j \leq m)$, in which all possible combinations of symbols appear equally often as rows in every $n \times t$ subarray. For the case $s_1 = s_2 = \cdots = s_m = s$, we use the symbol $OA(n, m, s, t)$.*

Definition 2 *A two-level compound orthogonal array of type (t_l, t_2, t_3), say A, is an $n_1 n_2 \times (m_1 + m_2)$ array with all factors at two levels, whose runs can be written as*

$$A = \begin{bmatrix} \mathbf{b}_1 & \\ \vdots & C_1 \\ \mathbf{b}_1 & \\ \vdots & \\ \mathbf{b}_{n_1} & \\ \vdots & C_{n_1} \\ \mathbf{b}_{n_1} & \end{bmatrix},$$

where the matrices

$$B = \begin{bmatrix} \mathbf{b}_1 \\ \vdots \\ \mathbf{b}_{n_1} \end{bmatrix} \quad and \quad C^* = \begin{bmatrix} \mathcal{C}_1 \\ \vdots \\ \mathcal{C}_{n_1} \end{bmatrix}$$

are $OA(n_1, m_1, 2, t_1)$ and $OA(n_1 \times n_2, m_2, 2, t_2)$ respectively. Moreover, for $1 \leq i \leq n_1$, C_i is an $OA(n_2, m_2, 2, t_{2i})$ and A is an $OA(n_1 \times n_2, (m_1 + m_2), 2, t_3)$, where $t_3 = min\{t_1, t_2\}$. The array A is denoted as $COA((n_1, n_2), (m_1, m_2), 2, (t_1, t_2, t_3))$.

Let $\mathcal{D}(n, 2^m, t)$ be a class of n-run designs involving m factors each at two levels such that any design belonging to this class is an $OA(n, m, 2, t)$, $t \geq 1$. Obviously, any design belonging to $\mathcal{D}(n, 2^m, t)$ will be a classical supersaturated design provided $t = 1$ and $n \leq m$.

Consider two designs $d_1 \in \mathcal{D}(n_1, 2^{m_1}, t_1)$ and $d_2 \in \mathcal{D}(n_2, 2^{m_2}, t_2)$ with design matrices $X^{(1)}$ and $X^{(2)}$ respectively, where

$$X^{(1)} = \left[\boldsymbol{x}_1^{(1)}, \boldsymbol{x}_2^{(1)}, \cdots, \boldsymbol{x}_{m_1}^{(1)}\right] \quad \text{and} \quad X^{(2)} = \left[\boldsymbol{x}_1^{(2)}, \boldsymbol{x}_2^{(2)}, \cdots, \boldsymbol{x}_{m_2}^{(2)}\right].$$

and $t_1, t_2 \geq 1$.

It is to be noted that the elements of the design matrices $X^{(1)}$ and $X^{(2)}$ are ± 1. Moreover, following the definition of the classes $\mathcal{D}(n_1, 2^{m_1}, t_1)$ and $\mathcal{D}(n_2, 2^{m_2}, t_2)$ of designs, it is clear that, for $1 \leq j \leq m_1$, $\left(\boldsymbol{x}_j^{(1)}\right)' \mathbf{1}_{n_1} = 0$ and, for $1 \leq j \leq m_2$, $\left(\boldsymbol{x}_j^{(2)}\right)' \mathbf{1}_{n_2} = 0$. Here, $\mathbf{1}_{n_1}$ and $\mathbf{1}_{n_2}$ are vectors of orders n_1 and n_2 with all elements equal to one. Define

$$s_{j_1 j_2}^{(10)} = \left(\boldsymbol{x}_{j_1}^{(1)}\right)' \boldsymbol{x}_{j_2}^{(1)},\ 1 \leq j_1 < j_2 \leq m_1, \quad \text{and} \quad s_{j_1 j_2}^{(01)} = \left(\boldsymbol{x}_{j_1}^{(2)}\right)' \boldsymbol{x}_{j_2}^{(2)},\ 1 \leq j_1 < j_2 \leq m_2.$$

For any $\boldsymbol{y}^{(1)} \in V^{(1)}$, let $n_{d_1}^{(1)}(\boldsymbol{y}^{(1)})$ be the number of times the level combination $\boldsymbol{y}^{(1)}$ appears in d_1 and n_{d_1} be the vector consisting of the elements $n_{d_1}^{(1)}(\boldsymbol{y}^{(1)})$, $\boldsymbol{y}^{(1)} \in V^{(1)}$. Similarly, for any $\boldsymbol{y}^{(2)} \in V^{(2)}$, let $n_{d_2}^{(2)}(\boldsymbol{y}^{(2)})$ be the number of times the level combination $\boldsymbol{y}^{(2)}$ appears in d_2 and n_{d_2} be the vector consisting of the elements $n_{d_2}^{(2)}(\boldsymbol{y}^{(2)})$, $\boldsymbol{y}^{(2)} \in V^{(2)}$. For $1 \le i_1 < i_2 \le n_1$, let $c_{i_1 i_2}^{(1)}$ be the number of coincidences of the levels of the m_1 factors between the i_1th and i_2th runs of the design d_1. Similarly, for $1 \le i_1 < i_2 \le n_2$, let $c_{i_1 i_2}^{(2)}$ be the number of coincidences of the levels of the m_2 factors between the i_1th and i_2th runs of the design d_2. Obviously, for $1 \le i \le n_1$, $c_{ii}^{(1)} = m_1$ and, for $1 \le i \le n_2$, $c_{ii}^{(2)} = m_2$.

3 Main Results

Consider a design $d_1 \in \mathcal{D}(n_1, 2^{m_1}, t_1), t_1 \ge 1$, with design matrix $X^{(1)}$, where

$$\begin{aligned} X^{(1)} &= \left[\boldsymbol{x}_1^{(1)}, \boldsymbol{x}_2^{(1)}, \cdots, \boldsymbol{x}_{m_1}^{(1)}\right], \text{ and} \\ d_1 &= \begin{bmatrix} \boldsymbol{y}_1^{(1)} \\ \boldsymbol{y}_2^{(1)} \\ \vdots \\ \boldsymbol{y}_{n_1}^{(1)} \end{bmatrix} = \begin{bmatrix} y_{11}^{(1)} y_{12}^{(1)} \cdots y_{1m_1}^{(1)} \\ y_{21}^{(1)} y_{22}^{(1)} \cdots y_{2m_1}^{(1)} \\ \vdots \\ y_{n_1 1}^{(1)} y_{n_1 2}^{(1)} \cdots y_{n_1 m_1}^{(1)} \end{bmatrix} \end{aligned}$$

For the case $t_1 = 1$, we assume that no two columns of the design matrix $X^{(1)}$ are aliased. For $1 \le k \le n_1$, let $d_{2k} \in \mathcal{D}(n_2, 2^{m_2}, t_{2k}), t_{2k} \ge 1$, with the design matrix

$$X^{(2k)} = \left[\boldsymbol{x}_1^{(2k)}, \boldsymbol{x}_2^{(2k)}, \cdots, \boldsymbol{x}_{m_2}^{(2k)}\right],$$

and

$$d_{2k} = \begin{bmatrix} \boldsymbol{y}_1^{(2k)} \\ \boldsymbol{y}_2^{(2k)} \\ \vdots \\ \boldsymbol{y}_{n_2}^{(2k)} \end{bmatrix} = \begin{bmatrix} y_{11}^{(2k)} y_{12}^{(2k)} \cdots y_{1m_2}^{(2k)} \\ y_{21}^{(2k)} y_{22}^{(2k)} \cdots y_{2m_2}^{(2k)} \\ \vdots \\ y_{n_2 1}^{(2k)} y_{n_2 2}^{(2k)} \cdots y_{n_2 m_2}^{(2k)} \end{bmatrix}$$

Define

$$d^* = \begin{bmatrix} d_{21} \\ d_{22} \\ \vdots \\ d_{2n_1} \end{bmatrix} \tag{1}$$

Let $\mathcal{D}^{(COA)}(n_1 \times n_2, 2^{m_1+m_2})$ be a class of $n_1 n_2$-run designs involving $(m_1 + m_2)$ factors each at two levels such that any design d belonging to the class $\mathcal{D}^{(COA)}(n_1 \times n_2, 2^{m_1+m_2})$ of designs is an $COA((n_1, n_2), (m_1, m_2), (t_1, t_2, t_3))$, where t_2 is the strength of the array d^* and

$t_3 = \min\{t_1, t_2\}$. Following Definition 2, we define d ($d \in \mathcal{D}^{(COA)}(n_1 \times n_2, 2^{m_1+m_2})$) as

$$d = \begin{bmatrix} \boldsymbol{y}_1^{(1)} & \\ \vdots & d_{21} \\ \boldsymbol{y}_1^{(1)} & \\ \vdots & \\ \boldsymbol{y}_{n_1}^{(1)} & \\ \vdots & d_{2n_1} \\ \boldsymbol{y}_{n_1}^{(1)} & \end{bmatrix}, \tag{2}$$

At this stage, it is to be noted that any design d belonging to $\mathcal{D}^{(COA)}(n_1 \times n_2, 2^{m_1+m_2})$ is a fractional factorial split-plot experiment involving m_1 whole-plot factors and m_2 subplot factors. Moreover, the design d is a supersaturated design at least with respect to the whole-plot factors. Consider any design d belonging to $\mathcal{D}^{(COA)}(n_1 \times n_2, 2^{m_1+m_2})$ with the design matrix X. Then X can be written as

$$\left.\begin{aligned} X &= [X_{WP},\ X_{SP}],\ X_{WP} = X^{(1)} \otimes \mathbf{1}_{n_2}, \quad \text{and} \\ X_{SP} &= \left[(X^{(21)})',\ (X^{(22)})', \cdots, (X^{(2n_1)})'\right]'. \end{aligned}\right\} \tag{3}$$

Here we assume no two columns of the design matrix X_{SP} are aliased. Following [3], here we define, as a measure of optimality of the design d,

$$E_{WP}(s^2) = \frac{n_2^2 \sum_{j_1=1}^{m_1} \sum_{j_2(\neq j_1)=1}^{m_1} (s_{j_1 j_2}^{(10)})^2}{m_1(m_1-1)}, \tag{4}$$

$$E_{SP}(s^2) = \frac{\sum_{j_1=1}^{m_2} \sum_{j_2(\neq j_1)=1}^{m_2} (\sum_{k=1}^{n_1} s_{j_1 j_2}^{(01k)})^2}{m_2(m_2-1)}, \tag{5}$$

where,

$$s_{j_1 j_2}^{(01k)} = \left(\boldsymbol{x}_{j_1}^{(2k)}\right)' \boldsymbol{x}_{j_2}^{(2k)},\ 1 \leq j_1 < j_2 \leq m_2,\ 1 \leq k \leq n_1,$$

and

$$E_{WSP}(s^2) = \frac{\sum_{j_1=1}^{m_1} \sum_{j_2=1}^{m_2} (\sum_{k=1}^{n_1} s_{j_1 j_2}^{(11k)})^2}{m_1 m_2}, \tag{6}$$

where,

$$s_{j_1 j_2}^{(11k)} = x_{j_1 k}^{(1)} (\mathbf{1}_{n_2})' \boldsymbol{x}_{j_2}^{(2k)},\ 1 \leq j_1 \leq m_1,\ 1 \leq j_2 \leq m_2,\ 1 \leq k \leq n_1.$$

Now, we define an overall measure of optimality of the split-plot design d as

$$E_d(s^2) = \frac{m_1(m_1-1)E_{WP}(s^2) + m_2(m_2-1)E_{SP}(s^2) + 2m_1 m_2 E_{WSP}(s^2)}{m(m-1)}, \tag{7}$$

where $m = (m_1 + m_2)$.

Remark 1 *It is to be remarked that, as per the construction of supersaturated split-plot design stated in (2), any two columns of the design matrix X one corresponding to a whole-plot factor and the other corresponding to a subplot factor forms an $OA(n_1 n_2, 2, 2, 2)$. This automatically implies, for $1 \leq j_1 \leq m_1,\ 1 \leq j_2 \leq m_2,\ 1 \leq k \leq n_1,\ s_{j_1 j_2}^{(11k)} = 0$.*

Remark 2 *It is to be remarked that if $t_1 \geq 2$ in (2), then $E_{WP}(s^2) = 0$. Similarly, if $t_2 \geq 2$, then $E_{SP}(s^2) = 0$. Finally, if $t_1, t_2 \geq 2$, then $E_d(s^2) = 0$.*

According to Remark 1, equation (7) can be rewritten as

$$E_d(s^2) = \frac{m_1(m_1-1)E_{WP}(s^2) + m_2(m_2-1)E_{SP}(s^2)}{m(m-1)}. \tag{8}$$

Corollary 1 *Consider the designs $d_1 \in \mathcal{D}(n_1, 2^{m_1}, t_1), t_1 \geq 1$, with design matrix $X^{(1)}$ and $d_2 \in \mathcal{D}(n_2, 2^{m_2}, t_2), t_2 \geq 1$, with design matrix $X^{(2)}$. Let*

$$X^{(1)} = \left[\boldsymbol{x}_1^{(1)}, \boldsymbol{x}_2^{(1)}, \cdots, \boldsymbol{x}_{m_1}^{(1)}\right], \qquad X^{(2)} = \left[\boldsymbol{x}_1^{(2)}, \boldsymbol{x}_2^{(2)}, \cdots, \boldsymbol{x}_{m_2}^{(2)}\right],$$

and

$$d_1 = \begin{bmatrix} \boldsymbol{y}_1^{(1)} \\ \boldsymbol{y}_2^{(1)} \\ \vdots \\ \boldsymbol{y}_{n_1}^{(1)} \end{bmatrix}, \qquad d_2 = \begin{bmatrix} \boldsymbol{y}_1^{(2)} \\ \boldsymbol{y}_2^{(2)} \\ \vdots \\ \boldsymbol{y}_{n_2}^{(2)} \end{bmatrix}$$

Following Definition 2, we construct a split-plot design $d \in \mathcal{D}_1^{(COA)}(n_1 \times n_2, 2^{m_1+m_2})$, based on designs d_1 and d_2, as follows

$$d = \begin{bmatrix} \boldsymbol{y}_1^{(1)} & \\ \vdots & d_2 \\ \boldsymbol{y}_1^{(1)} & \\ \vdots & \\ \boldsymbol{y}_{n_1}^{(1)} & \\ \vdots & d_2 \\ \boldsymbol{y}_{n_1}^{(1)} & \end{bmatrix}, \tag{9}$$

For any design d, constructed according to (9), we have

$$E_{WP}(s^2) = \frac{n_2^2 \sum_{j_1=1}^{m_1} \sum_{j_2(\neq j_1)=1}^{m_1} (s_{j_1 j_2}^{(10)})^2}{m_1(m_1-1)}$$

$$E_{SP}(s^2) = \frac{n_1^2 \sum_{j_1=1}^{m_2} \sum_{j_2(\neq j_1)=1}^{m_2} (s_{j_1 j_2}^{(01)})^2}{m_2(m_2-1)},$$

and the overall measure of optimality of the split-plot design d will be the same as given in (8).

Corollary 2 *Consider the designs* $d_1 \in \mathcal{D}(n_1, 2^{m_1}, t_1), t_1 \geq 1$, *with design matrix* $X^{(1)}$ *and* $d_2 \in \mathcal{D}(n_2, 2^{m_2}, \widetilde{t_2}), \widetilde{t_2} \geq 1$, *with design matrix* $X^{(2)}$. *Let*

$$X^{(1)} = \left[\boldsymbol{x}_1^{(1)}, \boldsymbol{x}_2^{(1)}, \cdots, \boldsymbol{x}_{m_1}^{(1)}\right], \qquad X^{(2)} = \left[\boldsymbol{x}_1^{(2)}, \boldsymbol{x}_2^{(2)}, \cdots, \boldsymbol{x}_{m_2}^{(2)}\right],$$

and

$$d_1 = \begin{bmatrix} \boldsymbol{y}_1^{(1)} \\ \boldsymbol{y}_2^{(1)} \\ \vdots \\ \boldsymbol{y}_{n_1}^{(1)} \end{bmatrix}, \qquad d_2 = \begin{bmatrix} \boldsymbol{y}_1^{(2)} \\ \boldsymbol{y}_2^{(2)} \\ \vdots \\ \boldsymbol{y}_{n_2}^{(2)} \end{bmatrix}$$

Let $\bar{d}_2$ *be a design obtained from the design* d_2 *through interchanging 0 and 1 and let* $\bar{X}^{(2)}$ *be the corresponding design matrix. Obviously,* $\bar{X}^{(2)} = -X^{(2)}$. *It is to be noted that, as per (1), here* $t_2(\geq 1)$ *is the strength of the array*

$$d^* = \begin{bmatrix} d_2 \\ \bar{d}_2 \end{bmatrix}$$

Following Definition 2, we construct a split-plot design $d \in \mathcal{D}_2^{(COA)}(n_1 \times n_2, 2^{m_1+m_2})$, *based on designs* d_1, d_2 *and* $\bar{d}_2$, *as follows*

$$d = \begin{bmatrix} \boldsymbol{y}_1^{(1)} & \\ \vdots & d_2 \\ \boldsymbol{y}_1^{(1)} & \\ \boldsymbol{y}_2^{(1)} & \\ \vdots & \bar{d}_2 \\ \boldsymbol{y}_2^{(1)} & \\ \vdots & \\ \boldsymbol{y}_{n_1-1}^{(1)} & \\ \vdots & d_2 \\ \boldsymbol{y}_{n_1-1}^{(1)} & \\ \boldsymbol{y}_{n_1}^{(1)} & \\ \vdots & \bar{d}_2 \\ \boldsymbol{y}_{n_1}^{(1)} & \end{bmatrix}, \tag{10}$$

For any design d, constructed according to (10), we have

$$E_{WP}(s^2) = \frac{n_2^2 \sum_{j_1=1}^{m_1} \sum_{j_2(\neq j_1)=1}^{m_1} (s_{j_1 j_2}^{(10)})^2}{m_1(m_1 - 1)}$$

$$E_{SP}(s^2) = \frac{n_1^2 \sum_{j_1=1}^{m_2} \sum_{j_2(\neq j_1)=1}^{m_2} (s_{j_1 j_2}^{(01)})^2}{m_2(m_2 - 1)},$$

and the overall measure of optimality of the split-plot design d will be the same as given in (8).

It is to be noted that, in (10), we are using the foldover technique for the subplot factors and thus we could obtain a more efficient design than that following (9), since this will improve the aliasing between the effects in the expense of a larger design. Let d_3 be an orthogonal array of strength 2 with the symbols 0 and 1 having $2n_2$ rows and (m_2+1) columns. Without loss of generality suppose the array d_3 is expressed as

$$d_3 = \begin{bmatrix} 0 & \\ \vdots & d_2 \\ 0 & \\ 1 & \\ \vdots & d_2^{**} \\ 1 & \end{bmatrix}$$

It is to be noted that both d_2 and d_2^{**} are $OA(n_2, m_2, 2, 1)$. Moreover, d_2 and d_2^{**} taken together constitutes and $OA(2n_2, m_2, 2, 2)$.

Corollary 3 *Consider the design $d_1 \in \mathcal{D}(n_1, 2^{m_1}, t_1), t_1 \geq 1$, with design matrix $X^{(1)}$ and the designs d_2, d_2^{**} with design matrices $X^{(2)}$, $X^{(2**)}$. Let*

$$\begin{aligned} X^{(1)} &= \left[\boldsymbol{x}_1^{(1)}, \boldsymbol{x}_2^{(1)}, \cdots, \boldsymbol{x}_{m_1}^{(1)}\right], \ X^{(2)} = \left[\boldsymbol{x}_1^{(2)}, \boldsymbol{x}_2^{(2)}, \cdots, \boldsymbol{x}_{m_2}^{(2)}\right], \\ X^{(2**)} &= \left[\boldsymbol{x}_1^{(2**)}, \boldsymbol{x}_2^{(2**)}, \cdots, \boldsymbol{x}_{m_2}^{(2**)}\right] \end{aligned}$$

and

$$d_1 = \begin{bmatrix} \boldsymbol{y}_1^{(1)} \\ \boldsymbol{y}_2^{(1)} \\ \vdots \\ \boldsymbol{y}_{n_1}^{(1)} \end{bmatrix}, \quad d_2 = \begin{bmatrix} \boldsymbol{y}_1^{(2)} \\ \boldsymbol{y}_2^{(2)} \\ \vdots \\ \boldsymbol{y}_{n_2}^{(2)} \end{bmatrix}, \ d_2^{**} = \begin{bmatrix} \boldsymbol{y}_1^{(2**)} \\ \boldsymbol{y}_2^{(2**)} \\ \vdots \\ \boldsymbol{y}_{n_2}^{(2**)} \end{bmatrix}$$

Following Definition 2, we construct a split-plot design $d \in \mathcal{D}_3^{(COA)}(n_1 \times n_2, 2^{m_1+m_2})$, *based on designs d_1, d_2 and d_2^{**}, as follows*

$$d \;=\; \left[\begin{array}{cc} \boldsymbol{y}_1^{(1)} & \\ \vdots & d_2 \\ \boldsymbol{y}_1^{(1)} & \\ \boldsymbol{y}_2^{(1)} & \\ \vdots & d_2^{**} \\ \boldsymbol{y}_2^{(1)} & \\ \vdots & \\ \boldsymbol{y}_{n_1-1}^{(1)} & \\ \vdots & d_2 \\ \boldsymbol{y}_{n_1-1}^{(1)} & \\ \boldsymbol{y}_{n_1}^{(1)} & \\ \vdots & d_2^{**} \\ \boldsymbol{y}_{n_1}^{(1)} & \end{array}\right], \tag{11}$$

It is to be noted that, as per (1), here $t_2(=2)$ is the strength of the array

$$d^* \;=\; \left[\begin{array}{c} d_2 \\ \bar{d}_2^{**} \end{array}\right]$$

For any design d, constructed according to (11), we have

$$E_{WP}(s^2) \;=\; \frac{n_2^2 \sum_{j_1=1}^{m_1} \sum_{j_2(\neq j_1)=1}^{m_1} (s_{j_1 j_2}^{(10)})^2}{m_1(m_1-1)},\; E_{SP}(s^2) \;=\; 0,$$

and the overall measure of optimality of the split-plot design d will be

$$E_d(s^2) \;=\; \frac{m_1(m_1-1)E_{WP}(s^2)}{m(m-1)}.$$

Remark 3 *It is to be noted that the supersaturated split-plot designs described in [11] can be obtained following Corollary 3.*

4 Derivation of Lower Bound

This section deals with the derivation of a lower bound of $E_d(s^2)$ as defined in (8). The following lemmas will be helpful in this regard.

Lemma 1 *For any design $d_1 \in \mathcal{D}(n_1, 2^{m_1}, t_1), t_1 \geq 1$, we have*

$$2\sum_{i_1=1}^{n_1} \sum_{i_2(\neq i_1)=1}^{n_1} c_{i_1 i_2}^{(1)} \;=\; n_1(n_1-2)m_1$$

Proof. Based on the design matrix $X^{(1)}$, we can write

$$X^{(1)}\left(X^{(1)}\right)' = \begin{pmatrix} m_1 & 2c_{12}^{(1)} - m_1 & \cdots & 2c_{1n_1}^{(1)} - m_1 \\ 2c_{12}^{(1)} - m_1 & m_1 & \cdots & 2c_{2n_1}^{(1)} - m_1 \\ \vdots & \vdots & \vdots & \vdots \\ 2c_{1n_1}^{(1)} - m_1 & 2c_{2n_1}^{(1)} - m_1 & \cdots & m_1 \end{pmatrix}$$

Now using the fact $1'_{n_1}X^{(1)} = 0$, we get the result immediately.

As per (1), define

$$d_2^* = \begin{pmatrix} d_{21} \\ \vdots \\ d_{2n_1} \end{pmatrix}$$

It is to be noted that $d_2^* \in \mathcal{D}(n_1n_2, 2^{m_2}, t_2)$ with $t_2 \geq 1$.

Lemma 2 *For any design $d_2^* \in \mathcal{D}(n_1n_2, 2^{m_2}, t_2)$, we have*

$$2\sum_{i_1=1}^{n_1n_2}\sum_{i_2(\neq i_1)=1}^{n_1n_2} c_{i_1i_2}^{(2*)} = n_1n_2(n_1n_2-2)m_2,$$

where $c_{i_1i_2}^{(2*)}$ is the number of coincidences of the levels of the m_2 factors between the i_1th and i_2th runs of the design d_2^*.

Lemma 3 *Let $f_1, f_2, \cdots, f_N$ be a set of nonnegative integers such that $\sum\limits_{i=1}^{n} f_i = c$. Then*

$$\sum_{i=1}^{n} f_i^2 \geq p_1w^2 + q_1(w+1)^2,$$

where w is the largest integer contained in c/n, p_1, q_1 are nonnegative integers such that $p_1 + q_1 = n$ and $p_1w + q_1(w+1) = c$.

Let $w_1^{(1)}$ be the largest integer contained in $n_1/4$ and $p_1^{(1)}, q_1^{(1)}$ are nonnegative integers such that $p_1^{(1)} + q_1^{(1)} = 4$ and $p_1^{(1)}w_1^{(1)} + q_1^{(1)}(1 + w_1^{(1)}) = n_1$. Define

$$\theta_1^{(1)} = p_1^{(1)}\left(w_1^{(1)}\right)^2 + q_1^{(1)}\left(1 + w_1^{(1)}\right)^2$$

Then we have the following lemma.

Lemma 4 *For any design $d_1 \in \mathcal{D}(n_1, 2^{m_1}, t_1), t_1 \geq 1$, we have*

$$\sum_{j_1=1}^{m_1}\sum_{j_2(\neq j_1)=1}^{m_1}\left(s_{j_1j_2}^{(10)}\right)^2 \geq m_1(m_1-1)\left(4\theta_1^{(1)} - n_1^2\right).$$

Proof. For $1 \le j_1 < j_2 \le m_1$, we have from definition,

$$\left(s^{(10)}_{j_1j_2}\right)^2 \;=\; \left(n^{j_1j_2}_{100} - n^{j_1j_2}_{101} - n^{j_1j_2}_{110} + n^{j_1j_2}_{111}\right)^2,$$

where $n^{j_1j_2}_{1\alpha\beta}$ is the number of times the j_1th and j_2th factors of d_1 appear at the levels α and β respectively, $\alpha, \beta = 0, 1$. It is then easy to note that

$$\begin{aligned}
4\sum_{\alpha=0}^{1}\sum_{\beta=0}^{1}\left(n^{j_1j_2}_{1\alpha\beta}\right)^2 &= \left(n^{j_1j_2}_{100} + n^{j_1j_2}_{101} + n^{j_1j_2}_{110} + n^{j_1j_2}_{111}\right)^2 + \left(n^{j_1j_2}_{100} + n^{j_1j_2}_{101} - n^{j_1j_2}_{110} - n^{j_1j_2}_{111}\right)^2 + \\
&\quad \left(n^{j_1j_2}_{100} - n^{j_1j_2}_{101} + n^{j_1j_2}_{110} - n^{j_1j_2}_{111}\right)^2 + \left(n^{j_1j_2}_{100} - n^{j_1j_2}_{101} - n^{j_1j_2}_{110} + n^{j_1j_2}_{111}\right)^2 \\
&= (s^{(10)}_{j_1j_2})^2 + \left(n^{j_1}_{10}\right)^2 + \left(n^{j_1}_{11}\right)^2 + \left(n^{j_2}_{10}\right)^2 + \left(n^{j_2}_{10}\right)^2 - n_1^2 = (s^{(10)}_{j_1j_2})^2 + n_1^2,
\end{aligned}$$

where $n^{j}_{1\alpha}$ is the number of times the jth factors of d_1 appear at the levels α, $1 \le j \le m_1$. Therefore,

$$\sum_{j_1=1}^{m_1}\sum_{j_2(\ne j_1)=1}^{m_1}\left(s^{(10)}_{j_1j_2}\right)^2 \;=\; 4\sum_{j_1=1}^{m_1}\sum_{j_2(\ne j_1)=1}^{m_1}\sum_{\alpha=0}^{1}\sum_{\beta=0}^{1}\left(n^{j_1j_2}_{1\alpha\beta}\right)^2 - m_1(m_1-1)n_1^2$$

Now, since $\sum_{\alpha=0}^{1}\sum_{\beta=0}^{1}\left(n^{j_1j_2}_{1\alpha\beta}\right) = n_1$, $1 \le j_1 < j_2 \le m_1$, and $n^{j_1j_2}_{1\alpha\beta} \ge 0$ for all $\alpha, \beta = 0, 1$, we have from Lemma 3,

$$\sum_{j_1=1}^{m_1}\sum_{j_2(\ne j_1)=1}^{m_1}\left(s^{(10)}_{j_1j_2}\right)^2 \;\ge\; m_1(m_1-1)\left(4\theta^{(1)}_1 - n_1^2\right)$$

Hence proved. Let $w^{(2)}_1$ be the largest integer contained in $n_1n_2/4$ and $p^{(2)}_1, q^{(2)}_1$ are nonnegative integers such that $p^{(2)}_1 + q^{(2)}_1 = 4$ and $p^{(2)}_1 w^{(2)}_1 + q^{(2)}_1(1 + w^{(2)}_1) = n_1n_2$. Define

$$\theta^{(2)}_1 = p^{(2)}_1\left(w^{(2)}_1\right)^2 + q^{(2)}_1\left(1 + w^{(2)}_1\right)^2$$

Then we have the following lemma.

Remark 4 *It is to be remarked that if $t_1 \ge 2$, then*

$$\sum_{j_1=1}^{m_1}\sum_{j_2(\ne j_1)=1}^{m_1}\left(s^{(10)}_{j_1j_2}\right)^2 \;=\; 0.$$

Lemma 5 *For any design $d_2^* \in \mathcal{D}(n_1n_2, 2^{m_2}, t_2)$, we have*

$$\sum_{j_1=1}^{m_2}\sum_{j_2(\ne j_1)=1}^{m_2}\left(\sum_{k=1}^{n_1} s^{(01k)}_{j_1j_2}\right)^2 \;\ge\; m_2(m_2-1)\left(4\theta^{(2)}_1 - n_1^2n_2^2\right).$$

Proof. The proof of this lemma follows along the line of the proof of Lemma 4.

Remark 5 *It is to be remarked that if $t_2 \geq 2$, then*

$$\sum_{j_1=1}^{m_2} \sum_{j_2(\neq j_1)=1}^{m_2} \left(\sum_{k=1}^{n_1} s_{j_1 j_2}^{(01k)} \right)^2 = 0.$$

Let $w_2^{(1)}$ be the largest integer contained in $(n_1-2)m_1/(2(n_1-1))$ and $p_2^{(1)}, q_2^{(1)}$ are nonnegative integers such that $p_2^{(1)} + q_2^{(1)} = n_1(n_1-1)$ and $p_2^{(1)} w_2^{(1)} + q_2^{(1)}(1+w_2^{(1)}) = n_1(n_1-2)m_1/2$. Define

$$\theta_2^{(1)} = p_2^{(1)} \left(w_2^{(1)}\right)^2 + q_2^{(1)} \left(1 + w_2^{(1)}\right)^2$$

Then we have the following lemma.

Lemma 6 *For any design $d_1 \in \mathcal{D}(n_1, 2^{m_1}, t_1), t_1 \geq 1,$, we have*

$$\sum_{j_1=1}^{m_1} \sum_{j_2(\neq j_1)=1}^{m_1} \left(s_{j_1 j_2}^{(10)}\right)^2 \geq 4\theta_2^{(1)} + m_1 n_1 (4m_1 - n_1 - m_1 n_1).$$

Proof. It is easy to observe that

$$\text{trace}\left[\left(X^{(1)}\left(X^{(1)}\right)'\right)^2\right] = \sum_{j_1=1}^{m_1} \sum_{j_2(\neq j_1)=1}^{m_1} \left(s_{j_1 j_2}^{(10)}\right)^2 + m_1 n_1^2, \tag{12}$$

and

$$\text{trace}\left[\left(X^{(1)}\left(X^{(1)}\right)'\right)^2\right] = m_1^2 n_1 + \sum_{i_1=1}^{n_1} \sum_{i_2(\neq i_1)=1}^{n_1} \left(2c_{i_1 i_2}^{(1)} - m_1\right)^2. \tag{13}$$

From equations (12), (13) and from Lemma 1, it follows that

$$\sum_{j_1=1}^{m_1} \sum_{j_2(\neq j_1)=1}^{m_1} \left(s_{j_1 j_2}^{(10)}\right)^2 = 4 \sum_{i_1=1}^{n_1} \sum_{i_2(\neq i_1)=1}^{n_1} \left(c_{i_1 i_2}^{(1)}\right)^2 + m_1 n_1 (4m_1 - n_1 - m_1 n_1). \tag{14}$$

The proof of Lemma 6 follows immediately from (14) and Lemma 3.

Let $w_2^{(2)}$ be the largest integer contained in $(n_1 n_2 - 2)m_2/(2(n_1 n_2 - 1))$ and $p_2^{(2)}, q_2^{(2)}$ are nonnegative integers such that $p_2^{(2)} + q_2^{(2)} = n_1 n_2 (n_1 n_2 - 1)$ and $p_2^{(2)} w_2^{(2)} + q_2^{(2)}(1 + w_2^{(2)}) = n_1 n_2 (n_1 n_2 - 2) m_2/2$. Define

$$\theta_2^{(2)} = p_2^{(2)} \left(w_2^{(2)}\right)^2 + q_2^{(2)} \left(1 + w_2^{(2)}\right)^2.$$

Then we have the following lemma.

Lemma 7 *For any design $d_2^* \in \mathcal{D}(n_1 n_2, 2^{m_2}, t_2), t_2 \geq 1$, we have*

$$\sum_{j_1=1}^{m_2} \sum_{j_2(\neq j_1)=1}^{m_2} \left(\sum_{k=1}^{n_1} s_{j_1 j_2}^{(01k)} \right)^2 \geq 4\theta_2^{(2)} + m_2 n_1 n_2 (4m_2 - n_1 n_2 - m_2 n_1 n_2).$$

Based on Lemma 4 and Lemma 6, let us define

$$\theta^{(1)} \;=\; \max\left\{m_1(m_1-1)\left(4\theta_1^{(1)}-n_1^2\right),\ 4\theta_2^{(1)}+m_1n_1(4m_1-n_1-m_1n_1)\right\} \tag{15}$$

Similarly, based on Lemma 5 and Lemma 7, let us define

$$\theta^{(2)} = \max\left\{m_2(m_2-1)\left(4\theta_1^{(2)}-n_1^2n_2^2\right),\ 4\theta_2^{(2)}+m_2n_1n_2(4m_2-n_1n_2-m_1n_1n_2)\right\} \tag{16}$$

From equations (8), (15)-(16), we present the following theorem which will serve as a benchmark for obtaining $E(s^2)$-optimal supersaturated split-plot design.

Theorem 1 *For any design $d \in \mathcal{D}^{(COA)}(n_1 \times n_2, 2^{m_1+m_2})$, we have*

$$E_d(s^2) \;\geq\; \frac{n_2^2\,\theta^{(1)}+\theta^{(2)}}{m(m-1)} \;=\; LB,\ (say).$$

Proof. From equation (4) and (15), it follows that

$$m_1(m_1-1)E_{WP}(s^2) \;=\; n_2^2\sum_{j_1=1}^{m_1}\sum_{j_2(\neq j_1)=1}^{m_1}(s_{j_1j_2}^{(10)})^2 \;\geq\; n_2^2\,\theta^{(1)}. \tag{17}$$

Again from equation (5) and (16), it follows that

$$m_2(m_2-1)E_{SP}(s^2) \;=\; \sum_{j_1=1}^{m_2}\sum_{j_2(\neq j_1)=1}^{m_2}\left(\sum_{k=1}^{n_1}s_{j_1j_2}^{(01k)}\right)^2 \;\geq\; \theta^{(2)}. \tag{18}$$

From Equations (17) and (18), the proof of Theorem 1 follows immediately.

Remark 6 *It is to be remarked that $E(s^2)$-optimal design belonging to $\mathcal{D}^{(COA)}(n_1 \times n_2, 2^{m_1+m_2})$ may not be $E(s^2)$-optimal over the class of designs $\mathcal{D}(n_1 \times n_2, 2^{m_1+m_2})$*

Remark 7 *For any design $d \in \mathcal{D}^{(COA)}(n_1 \times n_2, 2^{m_1+m_2})$ if we have $t_1 \geq 2$, then*

$$E_d(s^2) \;\geq\; \frac{\theta^{(2)}}{m(m-1)} \;=\; LB.$$

Remark 8 *For any design $d \in \mathcal{D}^{(COA)}(n_1 \times n_2, 2^{m_1+m_2})$ if we have $t_2 \geq 2$, then*

$$E_d(s^2) \;\geq\; \frac{n_2^2\,\theta^{(1)}}{m(m-1)} \;=\; LB.$$

To compare the efficiency of any design $d \in \mathcal{D}^{(COA)}(n_1 \times n_2, 2^{m_1+m_2})$, we define

$$Eff \;=\; \frac{LB}{E_d(s^2)} \tag{19}$$

5 Some illustrative examples

In this section we provide an example illustrating the idea that we have developed in this paper based compound orthogonal arrays.

Example 1 *Consider the design $d_1 \in \mathcal{D}(6, 2^{10}, 1)$ given below corresponding to the whole-plot factors.*

$$d_1 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

Also consider the following designs $d_{2k} \in \mathcal{D}(2, 2^{10}, 1)$ each corresponding to the subplot factors

$$d_{21} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix},\ d_{22} = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix},$$

$$d_{23} = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix},\ d_{24} = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \end{pmatrix},$$

$$d_{25} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \end{pmatrix},\ d_{26} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \end{pmatrix},$$

Now following the construction procedure described in (2), we get a design d belonging to $\mathcal{D}^{(COA)}(6 \times 2, 2^{(10+10)})$ is a supersaturated split-plot experiment involving 10 whole-plot factors and 10 subplot factors.
For this SSSPD, $LB = 3.7895$, while $E_d(s^2) = 7.7589$, hence the design is 50% efficient with respect to the proposed criterion. We note that in this design, the absolute value of the pair correlations between the whole-plot factors is equal to 4, the absolute value of the pair correlations between the subplot factors is equal to 4, while per construction the whole-plot factors are orthogonal to the subplot factors.

Example 2 *Consider the design $d_1 \in \mathcal{D}(6, 2^{10}, 1)$ given below corresponding to the whole-plot factors.*

$$d_1 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

Also consider the following designs $d_{2k} \in \mathcal{D}_2(4, 2^{11}, 1)$ each corresponding to the subplot factors

$$d_{21} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \; d_{22} = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix},$$

$$d_{23} = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \end{pmatrix}, \; d_{24} = \begin{pmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \end{pmatrix},$$

$$d_{25} = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{pmatrix}, \; d_{26} = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix},$$

Now following the construction procedure described in (2), we get a design d belonging to $\mathcal{D}^{(COA)}(6\times 4, 2^{(10+11)})$ is a supersaturated split-plot experiment involving 10 whole-plot factors and 11 subplot factors. We note that this design is a supersaturated design with respect to the whole-plot factors, since $t_1 = 1$ and $m_1 > n_1$.
This SSSPD is 100% efficient, since $E_d(s^2) = LB = 13.7143$. We note that in this design, the absolute value of the pair correlations between the whole-plot factors is equal to 8, while differently to the previous example, the split-plot factors are orthogonal to each other. Per construction the whole-plot factors are orthogonal to the subplot factors.

Remark 9 *It is to be remarked that, that the designs d_1 and d_{2j}, used in both examples were constructed randomly under the following characteristics:*

- *d_1 has strength t_1=1, $m_1 > n_1$ and no fully aliased columns,*
- *d_{2j}s have strength $t_2j \geq 1$ and they may include fully aliased columns. In the specific examples, $t_2j = 1$.*

Any design, from the literature of COA, appeared in the Introduction Section with the above characteristics could be used in a similar way.

6 Discussion

In this paper, a general combinatorial construction method for $E_d(s^2)$- efficient supersaturated split-plot designs is proposed. The method is based on the use of COA's of certain parameters. Specifically, for the construction form (2), for the whole-plot part of the design, a balanced initial design d_1 (strength t_1=1, and $m_1 > n_1$) with no fully aliased columns is needed, while for the split-plot part balanced designs d_{ij}s of strength $t_2j \geq 1$ should be used (fully-aliased columns could be included in these designs). The d_{ij}s taken together is desirable but not obligatory to result in an array of greater strength than their sub-parts. For example, in our second example, the d_{ij}s taken together resulted in an orthogonal array for the subplot effects, while the sub-designs included fully correlated columns. The final supersaturated split-plot design d^{COA} will have a strength equal to $t_3 = min(t_1, t_2) = 1$. The same comments are valid for the construction structures at the Corollaries, however there we need to use balanced designs d_{ij}s of strength $t_2j \geq 1$ in order not to have fully aliased columns at the final design.

The proposed method gives always designs in which the whole-plot factors will be orthogonal to the subplot factors. The method generalizes the [11] method, while we can calculate the $E_d(s^2)$-efficiency of these designs, using the lower bound proved above.

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